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Ex: Compute $\iint_R ye^{-xy}\,dA$ on $R=[0,2]\times[0,3]$

Sol 1: $\iint_R ye^{-xy}\,dA = \int_{x=0}^{2}\int_{y=0}^{3} ye^{-xy}\,dy\,dx$

Inner Integral: $\int_{y=0}^{3} ye^{-xy}\,dy$ $\quad\left(\begin{matrix} u=y & dv=e^{-xy}dy \\ du=dy & v=-\frac{1}{x}e^{-xy}\end{matrix}\right.$

$= \left[-\frac{y}{x}e^{-xy} - \int -\frac{1}{x}e^{-xy}\,dy\right]_{y=0}^{3}$

$= \left[-\frac{y}{x}e^{-xy} - \frac{1}{x^2}e^{-xy}\right]_{y=0}^{3}$

$= \left(-\frac{3}{x}e^{-3x} - \frac{1}{x^2}e^{-3x}\right) - \left(0 - \frac{1}{x^2}\right.$

$= e^{-3x}\left(-\frac{3}{x} - \frac{1}{x^2}\right) + \frac{1}{x^2}$

$\therefore \iint_R ye^{-xy}\,dA = \int_{x=0}^{2}\left(e^{-3x}\left(-\frac{3}{x}-\frac{1}{x^2}\right)+\frac{1}{x^2}\right)dx$

improper integral $\therefore$ abandon ship!

Sol 2: $\iint_R ye^{-xy}\,dA = \int_{y=0}^{3}\int_{x=0}^{2} ye^{-xy}\,dx\,dy$

Inner Integral: $\int_{x=0}^{2} ye^{-xy}\,dx = \int_{x=0}^{2} -e^{-xy}(-y)\,dx$

$= \int_{x=0}^{2} -e^{u}\,du = -e^{u}\Big|_{x=0}^{2} = -e^{-xy}\Big|_{x=0}^{2}$

$= \left(-e^{-2y}\right) - \left(-e^{-0y}\right) = 1 - e^{-2y}$

$= \int_{y=0}^{3}\left(1-e^{-2y}\right)dy = \left[y + \frac{1}{2}e^{-2y}\right]_{y=0}^{3}$

$= \left(3+\frac{1}{2}e^{-6}\right) - \left(0+\frac{1}{2}e^{0}\right) = \boxed{\frac{5}{2}+\frac{1}{2}e^{-6}}$

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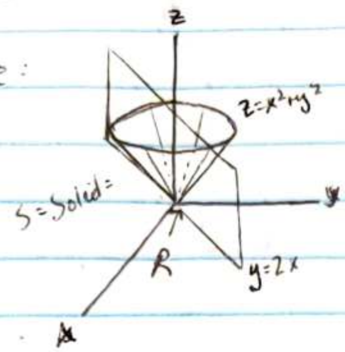
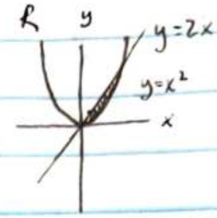
<u>GOAL</u>: Integrate over more complicated regions.

<u>Ex</u>: Compute net volume of the solid bounded by $z=x^2+y^2$, $y=2x$, $y=x^2$, $z=0$

Picture:





$x^2=2x$, so $x=0$ or $x=2$

$$Vol(S) = \iint_R ((x^2+y^2)-0)\,dA$$

$$= \int_{x=0}^{2} \int_{y=x^2}^{2x} (x^2+y^2)\,dy\,dx$$

$$= \int_{x=0}^{2} \left[x^2y + \frac{1}{3}y^3\right]_{x^2=y}^{2x} dx$$

$$= \int_{x=0}^{2} \left(\left(2x^3 + \frac{8}{3}x^3\right) - \left(x^4 + \frac{1}{3}x^6\right)\right) dx$$

$$= \int_{x=0}^{2} \left(\frac{14}{3}x^3 - x^4 - \frac{1}{3}x^6\right) dx$$

$$= \left[\frac{14}{12}x^4 - \frac{1}{5}x^5 - \frac{1}{21}x^7\right]_{x=0}^{2}$$

$$= \frac{7}{6}\cdot 16 - \frac{32}{5} - \frac{128}{21} - 0$$

$$= \frac{56}{3} - \frac{32}{5} - \frac{128}{21}$$

Take-Away: If R can be parameterized by
$R = \{(x,y) : c_1 \le x \le c_2,\ g_1(x) \le y \le g_2(x)\}$,

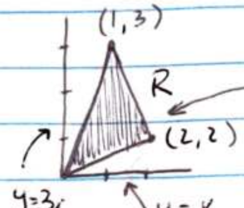
then... $\iint_R f(x,y)\,dA = \int_{x=c_1}^{c_2} \int_{y=g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$

Similarly, if R is parameterized by
$R = \{(x,y) : c_1 \le y \le c_2,\ g_1(y) \le x \le g_2(y)\}$

$$\iint_R f(x,y)\,dA = \int_{y=c_1}^{c_2} \int_{x=g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$$

Ex: Compute $\iint_R y\,dA$ over R, the triangle w/ vertices $(0,0), (1,3), (2,2)$.

Picture:



$R = \{(x,y) : 0 \le y \le 3\}$
$y - 2 = \left(\frac{2-3}{2-1}\right)(x-2)$
i.e. $y - 2 = -(x-2)$
i.e. $-(y-2) + 2 = x$
i.e. $4 - y = x$

$\therefore R = R_1 \cup R_2$
w/ $R_1 = \{(x,y) : 2 \le y \le 3,\ \frac{1}{3}y \le x \le 4-y\}$
$R_2 = \{(x,y) : 0 \le y \le 2,\ \frac{1}{3}y \le x \le y\}$

$\therefore \iint_R y\,dA = \iint_{R_1} y\,dA + \iint_{R_2} y\,dA.$

$$\iint_{R_1} y\,dA = \int_{y=2}^{3} \int_{x=\frac{1}{3}y}^{4-y} y\,dx\,dy$$
$$= \int_{y=2}^{3} y\,[x]_{x=\frac{1}{3}y}^{4-y}\,dy$$
$$= \int_{y=2}^{3} y\left(4 - \tfrac{4}{3}y\right)dy$$
$$= \int_{y=2}^{3} \left(4y - \tfrac{4}{3}y^2\right)dy$$

$$\left[2y^2 - \tfrac{4}{9}y^3\right]_{y=2}^{3}$$
$$= \left(2\cdot 9 - \tfrac{4}{9}\cdot 27\right) - \left(2\cdot 4 - \tfrac{4}{9}\cdot 8\right)$$
$$= (18 - 12) - \left(8 - \tfrac{32}{9}\right)$$
$$= -2 + \tfrac{32}{9} = \boxed{\tfrac{14}{9}}$$

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$$\iint_{R_2} y\,dA = \int_{y=0}^{2}\int_{x=\frac{1}{3}y}^{y} y\,dx\,dy$$

$$= \int_{y=0}^{2} y\,[x]_{x=\frac{1}{3}y}^{y}\,dy$$

$$= \int_{y=0}^{2} y\left(y-\tfrac{1}{3}y\right)dy$$

$$= \frac{2}{3}\int_{y=0}^{2} y^2\,dy$$

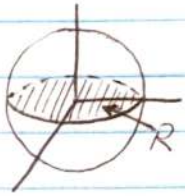
$$= \frac{2}{3}\cdot\frac{1}{3}y^3\Big|_{y=0}^{2}$$

$$= \frac{2}{9}(8-0) = \boxed{\frac{16}{9}}$$

$$\longrightarrow\ \therefore \iint_R y\,dA = \frac{14}{9}+\frac{16}{9} = \boxed{\frac{10}{3}}\ \blacksquare$$

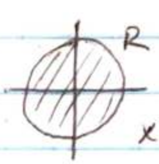
Motivating Question: What is the volume of a sphere?

Setup: $S = \{(x,y,z) : x^2+y^2+z^2 = r^2$



for $(x,y)\in R$:
$$z = \pm\sqrt{r^2-x^2-y^2}$$

$\therefore$ should integrate $Vol(S) = \iint_R 2\sqrt{r^2-x^2-y^2}\,dA$



$x^2+y^2=r^2$

$$\therefore R = \{(x,y) : -r \le x \le r, -\sqrt{r^2-x^2} \le y \le \sqrt{r^2-x^2}\}$$

$$Vol(S) = \int_{x=-r}^{r}\int_{y=-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \sqrt{r^2-x^2-y^2}\,dy\,dx$$

Exercise: Explain why that's terrible.
Evaluate.